THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2014–2015)
Introduction to Topology
Exercise 4 Continuity

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Do the exercises mentioned in lectures or in lecture notes.

2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be metric spaces. Can a function $f\colon X \to Y$ be continuous if $\mathcal{T}_Y$ is discrete?

3. Let $\mathbb{R}_{\ell\ell}$ be the real line with lower-limit topology (generated by $[a, b)$) and $\mathbb{R}$ be the standard real line. Give an example of continuous $f\colon \mathbb{R}_{\ell\ell} \to \mathbb{R}$ but it is not continuous when regarded as $\mathbb{R} \to \mathbb{R}$. Is there such an example for $\mathbb{R} \to \mathbb{R}_{\ell\ell}$?

4. Let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ and $\mathcal{S}_Y$ be a subbase for $\mathcal{T}_Y$. Is it true that f is continuous if and only if for every $S \in \mathcal{S}_Y$, $f^{-1}(S) \in \mathcal{T}_X$?

5. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The subspace (or induced or relative) topology on A is $\mathcal{T}|_A = \{ G \cap A : G \in \mathcal{T} \}$. Suppose $\mathcal{T}_A$ is another topology on A. Find a necessary and sufficient condition for the inclusion map $\iota\colon (A, \mathcal{T}_A) \to (X, \mathcal{T})$ such that $\mathcal{T}_A = \mathcal{T}|_A$.

6. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. The (finite) product topology on $X \times Y$ is
$$\mathcal{T}_{X\times Y} = \{ U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y \} .$$
Show that the projection mapping $\pi_X : X \times Y \to X$ is both open and continous.

7. Refer to the above product topology on $X \times Y$, show that a function $f\colon Y \to X \times Y$ is continuous if and only if both $\pi_X \circ f\colon Y \to X$ and $\pi_Y \circ f\colon Y \to X$ are continuous.

8. Let $X \times X$ be given the product topology of X. Show that $D = \{ (x, x) : x \in X \}$ as a subspace of $X \times X$ is homeomorphic to X.

9. Given a metric space (X, d), show that for each fixed $x_0 \in X$, the function
$$x \mapsto d(x, x_0) : X \to \mathbb{R}$$
is continuous.

10. Given a metric space (X, d), define a function $\rho : X^2 \times X^2 \to [0, \infty)$ by
$$\rho((x_1, x_2), (y_1, y_2)) = \max \{ d(x_1, y_1), d(x_2, y_2) \} .$$
It is known that ρ is a metric on X^2. Refer to this metric ρ, prove that the distance function $d\colon X \times X \to [0, \infty)$ is continuous.

11. Given a metric space (X, d) and a subset $A \subset X$, define $f\colon X \to [0, \infty)$ by $f(x) = \inf\{\, d(x, a) : a \in A \,\}$. Show that f is a continuous function.

12. Let $f, g\colon X \to \mathbb{R}$ be continuous functions. Prove that the following sets are respectively open and closed, $\{\, x \in X : f(x) < g(x) \,\}$, and $\{\, x \in X : f(x) \leq g(x) \,\}$.

    This can be generalized if $\mathbb{R}$ is replaced with Y of ordered topology.

13. Is it possible to find the following example? Justify your answer. Let $f\colon X \to Y$ be a continuous function between two metric spaces and B_k, $k \in \mathbb{N}$ be closed subsets in Y such that $\bigcup_{k=1}^{\infty} B_k$ is still a closed set. However, $\bigcup_{k=1}^{\infty} f^{-1}(B_k)$ is not closed in X.

14. Let $f\colon X \to Y$ be a continuous mapping. If $D \subset X$ is dense, is $f(D) \subset Y$ dense? What about the pre-image of a dense set?

15. Let $X = \bigcup_\alpha A_\alpha$ and each A_α be closed such that at every point $x \in X$, there is a neighborhood U of x that only intersects finitely many of A_α. Show that if each $f|_{A_\alpha}$ is continuous, then f is continuous on X.

    Remark. Such a family of A_α is called *locally finite*.

16. Apply the Tietz Extension Theorem (lecture version) to show that a continuous function $f\colon A \to \mathbb{R}$ on a closed subset of a metric space X can be extended to $\tilde{f}\colon X \to \mathbb{R}$.

    Hint. Note that $\mathbb{R}$ and $(-1, 1)$ are homeomorphic.

17. Is it possible to extend a continuous mapping $f\colon A \to \mathbb{R}^n$ on a closed subset of a metric space X? What if the target is $\mathbb{S}^n$?

18. Give an example of $f\colon \mathbb{R}^2 \setminus \{\,(0, 0)\,\} \to \mathbb{R}$ which cannot be extended to $\mathbb{R}^2$.

19. Give an example of $f\colon X \to Y$ which is 1-1 and continous but X is not homeomorphic to its image $f(X)$ as a subspace of Y.